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PATTERNS OF TRADE AND GROWTH UNDER INCREASING RETURNS: ESCAPE FROM THE POVERTY TRAP*

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We consider a two-sector model of intertemporal resource allocation in which the investment good sector exhibits an initial phase of increasing returns in production. The economy maximizes a discounted sum of one period utilities derived from the consumption good. If it is autarkic, it may face a poverty trap from which it cannot escape even if it follows an optimal policy. If it engages in trade with the outside world as a price taker, it may escape from the trap. The optimal patterns of production and trade are analysed for such an economy.

1. Introduction

Although convexity assumptions continue to play a dominant role in economic models capturing optimizing behavior, some attempts have been made to explore the implications of increasing returns in the context of optimal *intertemporal allocation*.[1] Indeed, when future utilities are discounted, a particularly interesting feature that distinguishes the aggregative optimal growth model with a Knightian "S-shaped" production function from that with a "classical" strictly concave Solow-type production function is the emergence of a critical stock below which the sequence of *optimal* capital stocks decreases to zero. In other words (depending on the discount factor) *there is an interval of capital stocks in which the economy is trapped if it is initially in it, even when it follows an optimal allocation policy over time*. The primary focus of this paper is to characterize the possibility of escaping from this *poverty trap* through "opening up" the economy.

A formal model of a small open economy over time is developed in Section 2. There are two goods: the "capital" or "investment" good (sector) has an S-shaped production function whereas the consumption good (sector) exhibits constant returns (both using the capital good as the sole input). The prices of these goods in the "international" markets are given to the economy (and, for simplicity, assumed

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1) We have in mind the collection of papers with the aggregative model by Clark (1971), Majumdar and Mitra (1982, 1983), Majumdar and Nermuth (1982), Dechert and Nishimura (1983) and Mitra and Ray (1984). A more recent paper by Mitra (1992) deals with optimal intertemporal allocation in the multisector model but focuses primarily on the 'undiscounted' or 'overtaking' optimality in the tradition of Ramsey, Weiszacker and David Gale. In this paper we shall restrict ourselves to the 'discounted' case. For descriptive models of "low income equilibrium trap" the reader should turn to Leibenstein (1954) and Nelson (1956).

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constant over time), and determine the income of the economy from the domestically produced stocks of the two goods. The allocation problems facing the economy in every period can be described as follows. First, it must decide on the fractions of income to be spent on "acquiring" the two goods. This decision determines whether the economy becomes a *net exporter or net importer* of a particular good. Next, it must decide how to divide between the two sectors the total capital good "acquired" (domestic production plus import or minus export). This decision determines the domestically produced stocks of the goods in the next period, and the story is repeated. The total consumption good "acquired" (domestic production plus import or minus export) generates "felicity" or utility according to an iso-elastic utility function.

Given a discount factor δ $(0<\delta<1)$, the economy attempts to make the sequence of allocation decisions so that the discounted sum of one-period utilities generated from consumptions is maximized.

To understand the role of trade, consider the case when the economy is autarkic. In the absence of trade the stocks of capital and consumption goods in any period are simply the domestically produced quantities. Let f be the S-shaped production function of the capital good sector. Then, as long as $f'(0)<1/\delta$, there is some interval $(0, y_c)$ such that any optimal programme starting from any initial stock in $(0, y_c)$ has the property that the optimal stocks decrease to zero.

It is clear that when the economy engages in trade, it has a larger class of feasible decisions. Let $\alpha>0$ be the output-capital ratio in the consumption good sector, and $p>0$ be the price of the capital good in terms of the consumption good. Then a unit of capital good can be used either in the further "direct" production of itself (by using the technology represented by f) or in an "indirect" acquisition of the capital good by first producing α units of the consumption good and exchanging it at the international market. The second route results in α/p units of the capital good. Once these two possibilities are kept in mind, it is not surprising that optimal allocations in the open economy depend crucially on the relation between α/p and the *average* productivity in the capital good industry (for an earlier analysis of the role of *average* productivity in characterizing efficient allocations in an aggregative economy with increasing returns, see Majumdar and Mitra (1982)).

To start with the extreme case, suppose that $\beta\equiv\alpha/p$ exceeds $f(b)/b$ where b is the point at which the average productivity of f, the investment good production function, attains its maximum. One can show that the optimal policy involves a complete specialization in the production of the consumption good (and "shutting down" the domestic capital good industry). Moreover, when $\delta\beta>1$, the economy can grow exponentially and enjoy unbounded consumption (an additional assumption is needed to ensure the existence of an optimal programme given our assumption of constant returns in the consumption good industry). Thus, by "opening up", the economy can take advantage of the favourable terms of trade (relative to the average productivity of the domestic capital good industry) and escape from the "poverty trap".

A complete analysis of the other case $(\beta<f(b)/b)$ requires some careful steps that fully exploit the simplifications introduced in the structure of the model. Here, again, the optimal capitals and consumptions increase to infinity. One can identify three stages through which a capital-poor economy will develop [see (15)]. In the first stage, when the initial capital belongs to an interval $(0, A)$, it does not use the

capital good industry at all, but produces (and exports) the consumption good only. Next, the optimal programme displays a reversal of trade pattern as the economy moves beyond the critical stock "A". In this phase, the economy does not use the consumption good industry at all: all the input is allocated for the production of the capital good which is also exported. At a higher level of initial stock "B" ($> A$), it becomes optimal to use both industries, although for a while the capital good is exported. Finally, there is a threshold where both the industries are used, but the capital good industry reaches a maximum size. The consumption good industry continues to grow (given our constant returns assumption!), and it becomes the export good for "financing" (importing) growing needs for capital inputs.

Thus, our framework is able to provide sufficient conditions for escaping the poverty trap, and leads to a complete classification of optimal patterns of trade and growth.[2)]

From a broader perspective, this paper is linked to the literature on optimal capital accumulation in an open economy as well as to the discussions on gains from trade in a dynamic economy. We make no attempt to review the vast related literature (see Kemp and Wan (1993)) but do wish to emphasize that our analysis is explicitly dynamic and not oriented towards a study of steady states only (see, e.g., Srinivasan (1989)).

It should be stressed that we make some drastic simplifications that have enabled us to keep the mathematical arguments short and simple. We believe that progress can be made in several directions, perhaps at the cost of more involved and complicated analysis. It will be useful to allow for (a) a durable capital good and (b) a strictly concave production function ("strongly productive" in the sense of Gale and Sutherland (1968)) in the consumption good sector. The assumption that the terms of trade do not change over time should also be relaxed and the implications of an exogenously given sequence of time-dependent international prices ought to be explored. Our "small country" assumption also limits the scope of this paper, but we are not sure how the patterns of trade and growth will evolve with increasing returns in a more general market structure.

2. The framework

2.1 Formal description

Formally, the framework is described by the objects $(f, g, \delta, w, \boldsymbol{k}, p)$ where f and g are functions from $\mathfrak{R}_+$ to $\mathfrak{R}_+$, $0 < \delta < 1$, w is a function from $\mathfrak{R}_+$ to $\mathfrak{R}$, $\boldsymbol{k} \geqslant 0$ and $p > 0$.

A *programme* from $\boldsymbol{k}$ is a non-negative sequence $\{k_t, x_t, c_{t+1}\}$ such that

$$k_0 = \boldsymbol{k},\ 0 \leqslant x_t \leqslant k_t,\ 0 \leqslant c_{t+1};\ pk_{t+1} + c_{t+1} = pf(x_t) + g(k_t - x_t) \text{ for } t \geqslant 0 \quad (1)$$

An *autarkic programme* from $\boldsymbol{k}$ is a programme $\{k_t, x_t, c_{t+1}\}$ from $\boldsymbol{k}$ which

2) Strictly speaking, there is a 'degenerate' or 'knife-edge' possibility in which a multiplicity of optimal policies makes it impossible to predict the patterns of production and trade. See Case (iii) in Section 4.

satisfies

$$c_{t+1} = g(k_t - x_t) \quad \text{for } t \geqslant 0 \tag{2}$$

A programme $\{k_t^*, x_t^*, c_{t+1}^*\}$ from $\boldsymbol{k}$ is called an *optimal programme* from $\boldsymbol{k}$ if

$$\sum_{t=0}^{\infty} \delta^t w(c_{t+1}^*) \geqslant \sum_{t=0}^{\infty} \delta^t w(c_{t+1}) \tag{3}$$

for every programme $\{k_t, x_t, c_{t+1}\}$ from $\boldsymbol{k}$. Similarly, an autarkic programme $\{\hat{k}_t, \hat{x}_t, \hat{c}_{t+1}\}$ from $\boldsymbol{k}$ is called an *optimal autarkic programme* if

$$\sum_{t=0}^{\infty} \delta^t w(\hat{c}_{t+1}) \geqslant \sum_{t=0}^{\infty} \delta^t w(c_{t+1}) \tag{4}$$

for every autarkic programme $\{k_t, x_t, c_{t+1}\}$ from $\boldsymbol{k}$.

2.2 Interpretation

The model outlined above is to be interpreted as a simple framework in which one can contrast the optimal growth and trade patterns of a "small" country under free trade (at constant world prices) with those under an autarkic regime.

To elaborate, consider a small country over time which faces *fixed* "world" or international prices for a "consumption" and a "capital" or "investment" good. Let $p > 0$ denote the relative price of the investment good (in terms of the consumption good), assumed constant for all periods.

The production function of the capital (respectively, consumption) good "industry" or "sector" is given by f (respectively, g). The capital good is assumed to depreciate fully within a period. The country initially (i.e., in period 0) has a capital stock $k_0 = \boldsymbol{k}$. It is allocated between the two sectors in non-negative quantities: x_0 is the capital used in the investment good sector and $k_0 - x_0$ is the capital used in the consumption good sector. As a result, in the next period (i.e., period 1), the country starts with stocks in the amounts $f(x_0)$ of the capital good and $g(k_0 - x_0)$ of the consumer good. At the international prices, the *income* i_1 of the country in period 1 is, therefore,

$$i_1 = pf(x_0) + g(k_0 - x_0). \tag{5}$$

This income is spent on acquiring any non-negative quantities (k_1, c_1) of the two goods satisfying

$$pk_1 + c_1 = i_1. \tag{6}$$

Of course, if $k_1 > f(x_0)$, the country is a *net* importer of the capital good whereas if $k_1 < f(x_0)$, it is a net exporter of that good. Similarly, the choice of c_1 determines whether the consumption good is exported or imported. And, writing

$$p[k_1 - f(x_0)] + [c_1 - g(k_0 - x_0)] = 0. \tag{7}$$

We see that we require that the value of exports must equal the value of imports (a "balance of trade" condition).

Now, the consumption good generates a utility $w(c_1)$ according to the one-period felicity or welfare function w. The capital good k_1 must be allocated as an input between the two sectors, and this allocation determines the domestic productions of the two goods as well as the income i_2 in period 2, and the story is repeated. Thus, a programme is a complete specification of the sequence of decisions on the allocation $(x_t, k_t - x_t)$ of the capital (k_t) between the two sectors as well as the decisions on spending the available income to acquire the two goods (k_{t+1}, c_{t+1}). As a result of these decisions, (and the condition that the value of exports must equal the value of imports), the pattern of trade in each period is also completely specified.

If a country is *not* allowed to trade, it has to consume the domestic production of the consumption good (so that $c_{t+1} = g(k_t - x_t)$) and this also means that k_{t+1} the stock of capital in period $t+1$, equals $f(x_t)$, the quantity that is domestically produced as a result of using x_t in the capital good industry. This description corresponds to our formal definition of an autarkic program.

A discount factor $0 < \delta < 1$ is given; the objective of the country (either viewed as an "open" economy engaged in trade or a "closed" autarkic economy) is to maximize the discounted sum of one-period felicities obtained from consumption. This corresponds to our formal definition of optimal and autarkic optimal programmes.

2.3 Technology and preferences: assumptions and simplifications

We would like to capture in our framework the feature that the production function in the investment good sector is subject to increasing returns for low input levels, and diminishing returns for high input levels. On the other hand, the production function in the consumption good sector as well as the welfare function are of the standard type (often called "classical") used in optimal growth theory. These features are made explicit in the following assumptions:

(F.1) $f(0) = 0$; *f is twice continuously differentiable on* $\Re_+$, *with* $f'(x) > 0$ *for* $x \geqslant 0$.
(F.2) *f satisfies the end-point conditions*: $\lim_{x\to\infty} f'(x) = 0$; $\lim_{x\to 0} f'(x) > 1$.
(F.3) *There is a scalar, a, such that* (i) $0 < a < \infty$; (ii) $f''(x) > 0$ *for* $0 \leqslant x < a$; (iii) $f''(x) < 0$ *for* $x > a$.
(G) *There is* $\alpha > 0$ *such that* $g(x) = \alpha x$ *for all* $x \geqslant 0$.
(W) *There is* $v \in (0, 1)$ *such that* $w(c) = c^{1-v}$.

As we noted earlier, the assumption (G) of constant returns in the consumption good sector and the functional form (W) specified for the welfare function enable us to obtain our sharp results with a minimal amount of algebra.

The production function f in the investment good sector exhibits an initial phase of increasing returns. To analyse the implications, we define the "average product" function h (see Majumdar and Mitra (1982)) as follows:

$$h(x) = [f(x)/x] \text{ for } x > 0;\ h(0) = \lim_{x\to 0}[f(x)/x].$$

Under our assumptions, it is easily checked that $h(0) = f'(0)$. Furthermore, there exist uniquely determined numbers b, $\bar{k}$ satisfying (i) $0 < a < b < \bar{k} < \infty$; (ii) $f'(b) = h(b)$; (iii) $f(\bar{k}) = \bar{k}$. It can be verified that for $0 < x < \bar{k}$, $x < f(x) < \bar{k}$, and for $x > \bar{k}$, $\bar{k} < f(x) < x$; for $0 < x < b$, $f'(x) > h(x)$ and h is increasing, while for $x > b$, $f'(x) < h(x)$ and h is decreasing; for $0 < x < a$, $f'(x)$ is increasing, while for

$x > a$, $f'(x)$ is decreasing. The functions, f and h, together with the numbers a, b, $\bar{k}$ may be represented diagrammatically as shown in Figure 1.

3. Optimal accumulation under autarky: "the poverty trap"

Consider, first, the situation in which all intertemporal choices are restricted to autarkic programmes: our economy is *closed* and is *not* allowed to trade. Here, to highlight the implications of increasing returns we shall characterize optimal autarkic programmes when the discount factor δ satisfies $f'(0) < 1/\delta$. From the earlier studies of Majumdar and Mitra (1982) and Dechert and Nishimura (1983), we know that it is in this range of values of the discount factor that one may expect a striking difference in the behaviour of optimal programmes between the "classical" (convex) and non-classical environments. Keeping this in mind, we find it useful to separate the analysis into three cases:

Case (i) $\delta f'(a) \leq 1$,
Case (ii) $\delta h(b) < 1 < \delta f'(a)$,
Case (iii) $\delta f'(0) < 1 \leq \delta h(b)$.

We do not discuss the case $1/\delta \leq f'(0)$ at all as this case does not lead to the emergence of a "poverty trap" that we explore in this paper (see Majumdar and Mitra (1982)).

For the autarkic economy, the problem of intertemporal optimization can be written as:

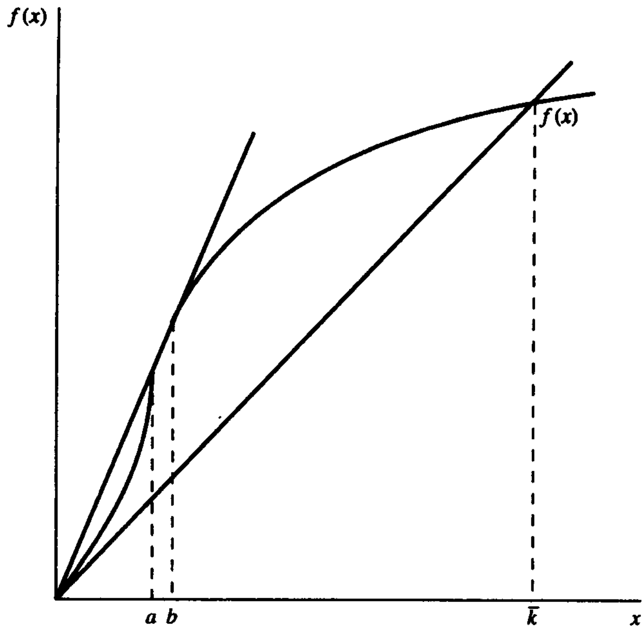
$$\begin{aligned} &\text{Maximize} \quad \sum_{t=0}^{\infty} \delta^t w[\alpha(k_t - x_t)] \\ &\text{subject to} \quad k_0 = \boldsymbol{k} \qquad\qquad (Q') \\ &\qquad 0 \leq x_t \leq k_t \quad \text{for } t \geq 0 \\ &\qquad f(x_t) = k_{t+1} \quad \text{for } t \geq 0. \end{aligned}$$

A compactness argument can be used to show that the optimization problem (Q') does have a solution for every fixed initial $\boldsymbol{k} \geq 0$. The long run behaviour of optimal programmes can be precisely characterized. Let $\{\hat{k}_t, \hat{x}_t\}$ be any solution. Denote $f^{-1}(\boldsymbol{k})$ by $\boldsymbol{y}$ and simply write $x_{-1} \equiv \boldsymbol{y}$. Furthermore, associate with each sequence $\{k_t, x_t\}$ satisfying the inequalities in (Q') the sequence $\{y_t\}$ defined by $y_t = x_{t-1}$ for $t \geq 0$. Then

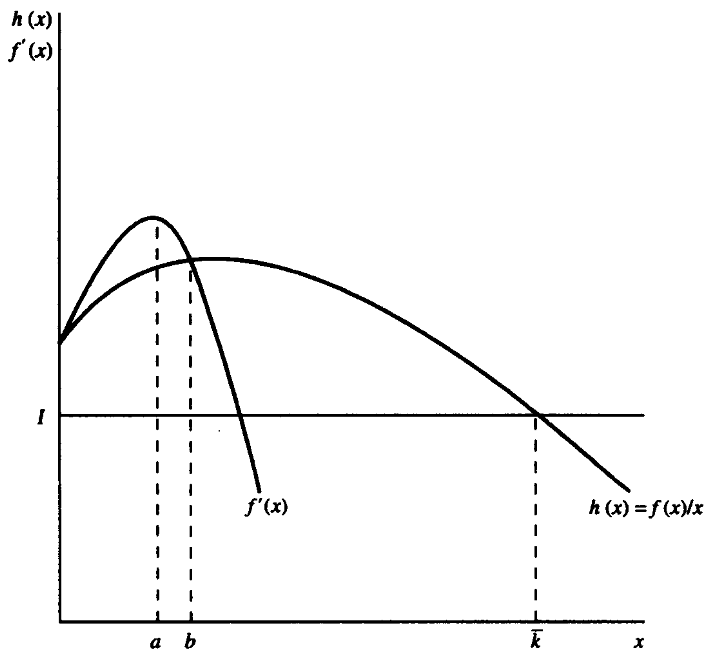
$$0 \leq k_t - x_t = f(x_{t-1}) - x_t = f(y_t) - y_{t+1} \quad \text{for } t \geq 0$$

So $\{\hat{y}_t\}$ solves the optimization problem:

$$\begin{aligned} &\text{Maximize} \quad \sum_{t=0}^{\infty} \delta^t \alpha^{1-v}[f(y_t) - y_{t+1}]^{1-v} \\ &\text{subject to} \quad y_0 = \boldsymbol{y} \qquad\qquad (Q) \\ &\qquad 0 \leq y_{t+1} \leq f(y_t) \quad \text{for } t \geq 0. \end{aligned}$$



Production function



Marginal and average
product functions

FIGURE 1.

The analysis of the long run behaviour of any $\{\hat{y}_t\}$ solving (Q) can now be completed by using the earlier studies of dynamic optimization with increasing returns in an aggregative framework.

Case (i): If $\delta f'(a) < 1$, then we can use Theorem 5.1 in Majumdar and Mitra (1982) to conclude that $\hat{y}_t$ and $\hat{c}_t$ decrease to zero as $t \to \infty$. If $\delta f'(a) = 1$, then $\delta f'(x) < 1$ for all $x \neq a$, and we can show that for all $0 < y < a$, $\hat{y}_t$ and $\hat{c}_t$ decrease to zero as $t \to \infty$. To check this last assertion, note that by Corollary 1 of Dechert and Nishimura (1983) we have either (a) $\hat{y}_{t+1} \geqslant \hat{y}_t$ for all $t \geqslant 0$, or (b) $\hat{y}_{t+1} \leqslant \hat{y}_t$ for all $t \geqslant 0$. Furthermore, by Theorem 2 of Dechert and Nishimura (1983), we must have $\hat{y}_t$ converging to a steady-state (that is, to the origin or a solution of $\delta f'(k) = 1$). Thus, in sub-case (a), $\hat{y}_t$ must converge to a.

Denote $f(\hat{y}_t) - \hat{y}_{t+1}$ by m_{t+1} for $t \geqslant 0$, and the function $\alpha^{1-v} m^{1-v}$ by $u(m)$ for $m \geqslant 0$. By the Ramsey-Euler equations, we have $u'(m_t) = \delta u'(m_{t+1}) f'(\hat{y}_t)$ for $t \geqslant 1$. Since $\delta f'(\hat{y}_t) \leqslant 1$ for $t \geqslant 1$, we have $u'(m_t) \leqslant u'(m_{t+1})$ and so $m_t \geqslant m_{t+1}$. Thus, m_t converges to $f(a) - a$, and $m_1 = f(\hat{y}_0) - \hat{y}_1 \geqslant f(a) - a$. But $f(\hat{y}_0) - \hat{y}_1 \leqslant f(\hat{y}_0) - \hat{y}_0$ (since $\hat{y}_1 \geqslant \hat{y}_0$) $< f(a) - a$ (since $\hat{y}_0 < a$ and $f'(x) > 1$ for all $0 \leqslant x \leqslant a$), a contradiction which rules out sub-case (a).

Thus, we have sub-case (b), and so $\hat{y}_t \leqslant \hat{y}_0 < a$ for all $t \geqslant 0$. Thus, $\delta f'(\hat{y}_t) \leqslant f'(\hat{y}_0) < 1$ for all $t \geqslant 0$, so $\hat{y}_t$ cannot converge to a solution of $\delta f'(k) = 1$. So, $\hat{y}_t$ converges to zero, which implies, in sub-case (b) that $\hat{y}_t$ decreases to zero. By the Ramsey-Euler equations, m_t (and hence $\hat{c}_t$) must decrease to zero.

Case (ii): In this case, by Theorem 5 (and Corollary 1) of Dechert and Nishimura (1983), there is a critical stock $y_c > 0$, such that if $0 < \hat{y}_0 < y_c$, then along any optimal programme, y_t decreases to zero.

Case (iii): Here, there exist two positive solutions to the equation $\delta f'(k) = 1$; call them y_* and y^*, with $0 < y_* < y^*$. Also, there exists $y_* < \tilde{y} \leqslant y^*$ such that $\delta h(\tilde{y}) = 1$. By Theorem 4 (and Corollary 1) of Dechert and Nishimura (1983), there is a critical stock y_c satisfying $0 < y_c < \tilde{y}$ such that if $0 < y_0 < y_c$, then along any optimal programme, $\hat{y}_t$ decreases to zero.

To summarize: *so long as* $\delta f'(0) < 1$, *there is always a critical stock* $y_c > 0$, *such that if* $0 < y < y_c$, *then for an optimal programme from* y, *we must have* $\hat{y}_t$ *decreasing to zero*.

It is worth remembering that the existence of this interval $(0, y_c)$ that we call a *poverty trap* is compatible with the existence of (feasible) programmes along which capital stocks do *not* decumulate to zero (i.e., with the assumption $f'(0) > 1$). This, incidentally, could not arise in the "classical" environment of a strictly concave f where either there was no poverty trap or all economies were in such a trap.

4. Escaping the poverty trap: trade as an engine of growth

We now proceed to examine systematically how trade can overcome a "poverty trap". A moment's reflection will convince the reader that the growth possibilities of the *open* economy depend on the terms of trade that it faces; and, whether trade provides a more effective way of accumulating capital than the domestic investment good sector can allow. To make the conditions precise, write

$$\beta \equiv \alpha/p \tag{8}$$

and assume throughout this section:

(E.1) $\quad$ (i) $f'(0) < 1/\delta$ and (ii) $\delta\beta > 1$. $\quad$ (9)

Notice that under (E.1), $\beta > 1/\delta > f'(0)$, captures the idea that trade provides a more effective way of capital accumulation than the domestic investment good sector at least for sufficiently low levels of capital.

4.1 Intertemporal optimization in the open economy

Going back to our open economy of Section 2, we see that the intertemporal optimization exercise that the open economy faces can be written as the following problem (P):

$$\begin{aligned} &\text{Maximize} && \sum_{t=0}^{\infty} \delta^t w[\alpha(k_t - x_t) + pf(x_t) - pk_{t+1}] \\ &\text{subject to} && k_0 = \boldsymbol{k} \\ &&& 0 \leqslant x_t \leqslant k_t \quad \text{for } t \geqslant 0 \\ &&& \alpha(k_t - x_t) + pf(x_t) \geqslant pk_{t+1} \quad \text{for } t \geqslant 0. \end{aligned} \tag{P}$$

Notice that x_t affects only the t-th term of the objective function. Given k_t, it is clear that x_t must be chosen so as to solve the following maximization problem (R′):

$$\begin{aligned} &\text{Maximize } \alpha(k_t - x_t) + pf(x_t) \\ &\text{subject to } 0 \leqslant x_t \leqslant k_t. \end{aligned} \tag{R'}$$

Recall (Figure 1) that b is the unique input level at which the (average product) function h (of the investment good sector) attains its maximum. We break up our analysis into three cases, one of which is a "knife-edge" or "degenerate" possibility.

Case (i): $\beta > h(b)$

In this case, we shall make the assumption

(E.2) $\quad \delta\beta^{1-v} < 1$

in order to avoid the problem of non-existence of optimal programmes (with constant returns in the consumption good sector and constant terms of trade). Write

$$\alpha(k_t - x_t) + pf(x_t) \equiv \alpha k_t + [h(x_t) - \beta]px_t \tag{10}$$

and observe that $h(x) < \beta$ for all x. Hence, the constrained maximization problem (R′) has a unique solution $\hat{x}_t = 0$. Thus, our optimization problem (P) is reduced to:

$$\begin{aligned} &\text{Maximize} && \sum_{t=0}^{\infty} \delta^t [\alpha k_t - pk_{t+1}]^{1-v} \\ &\text{subject to} && k_0 = \boldsymbol{k} \\ &&& \alpha k_t \geqslant pk_{t+1} \geqslant 0 \quad \text{for all } t \geqslant 0. \end{aligned}$$

This, in turn, can be further reduced to the following dynamic optimization problem (P′):

$$\begin{aligned}&\text{Maximize} && \sum_{t=0}^{\infty} \delta^t[\beta k_t - k_{t+1}]^{1-v} \\ &\text{subject to} && k_0 = \boldsymbol{k} \qquad\qquad\qquad (\mathrm{P}') \\ & && \beta k_t \geqslant k_{t+1} \geqslant 0 \quad \text{for } t \geqslant 0\end{aligned}$$

This last form (P′) of the problem is a familiar one. It is mathematically equivalent to an optimal growth problem for a non-trading economy, which has a linear technology with output-capital ratio β, a utility function c^{1-v}, and a discount factor $0 < \delta < 1$.

The solution to this problem is well-known. Briefly, given $\delta\beta^{1-v} < 1$ there is a unique optimal programme from every initial stock $\boldsymbol{k}$. Thus, there is an optimal transition function, $\mathfrak{F}: \mathfrak{R}_+ \to \mathfrak{R}_+$, such that if $\{\hat{k}_t\}$ is the optimal programme from $\boldsymbol{k}$, then

$$\hat{k}_{t+1} = \mathfrak{F}(\hat{k}_t) \quad \text{for } t \geqslant 0. \tag{11}$$

In fact, for the above specification, one can solve for $\mathfrak{F}$ explicitly. It is linear in its argument:

$$\mathfrak{F}(k) = \lambda k \quad \text{for } k \geqslant 0$$

To find λ, write the appropriate Ramsey-Euler equation:

$$(1 - v)[\beta k - \mathfrak{F}(k)]^{-v} = \delta\beta(1 - v)[\beta\mathfrak{F}(k) - \mathfrak{F}(\mathfrak{F}(k))]^{-v}.$$

Substitute the linear form $\mathfrak{F}(k) = \lambda k$ (with λ as yet undetermined) to obtain

$$[\beta k - \lambda k]^{-v} = \delta\beta[\beta\lambda k - \lambda^2 k]^{-v}.$$

Thus, $\lambda = (\delta\beta)^{1/v}$ solves the above equation. Now, $(\delta\beta)^{1/v}/\beta = \delta^{1/v}\beta^{(1-v)/v} = [\delta\beta^{1-v}]^{1/v} < 1$, so that $\lambda < \beta$. To formally complete the demonstration that the optimal transition function is as described above, simply define the function

$$\mathfrak{F}(k) = (\delta\beta)^{1/v}k \quad \text{for } k \geqslant 0. \tag{12}$$

Further, given any $k > 0$, define $\{\hat{k}_t\}$ by $\hat{k}_0 = k$, and $\hat{k}_{t+1} = \mathfrak{F}(\hat{k}_t)$ for $t \geqslant 0$. Since $(\delta\beta)^{1/v} < \beta$, $\{\hat{k}_t\}$ will satisfy the constraints of (P′). Then, by essentially repeating the above calculations in reverse order, the sequence $\{\hat{k}_t\}$ can be shown to satisfy the Ramsey-Euler equations corresponding to problem (P′). Finally, the appropriate transversality condition is satisfied. To check this, note that

$$\hat{p}_t\hat{k}_t = \delta^t(1 - v)[\beta\hat{k}_t - \hat{k}_{t+1}]^{-v}\hat{k}_t = \delta^t(1 - v)[\beta - (\delta\beta)^{1/v}]^{-v}\hat{k}_t^{1-v}.$$

Now, $\delta^t\hat{k}_t^{1-v} = [\delta(\delta\beta)^{(1-v)/v}]^t k^{1-v}$ so that $\delta^t\hat{k}_t^{1-v} \to 0$ as $t \to \infty$, since $0 < \delta(\delta\beta)^{(1-v)/v} = \delta^{1/v}\beta^{(1-v)/v} = [\delta\beta^{1-v}]^{1/v} < 1$. Thus, $\hat{p}_t\hat{k}_t \to 0$ as $t \to \infty$. This completes the demonstration that the optimal transition function is given by

$$\mathfrak{F}(k) = (\delta\beta)^{1/v}k \quad \text{for } k \geqslant 0.$$

Since $(\delta\beta)^{1/v} > 1$ by assumption (E.1), the optimal programme $\{\hat{k}_t\}$ from any initial stock $\boldsymbol{k} > 0$ has the property that $\hat{k}_t$ increases (exponentially) to infinity as $t \to \infty$. The growth factor of the economy, along the optimal programme, is constant since

$$(\hat{k}_{t+1}/\hat{k}_t) = (\delta\beta)^{1/v} \quad \text{for } t \geq 0. \tag{13}$$

The optimal consumption sequence $\{\hat{c}_{t+1}\}$ is given by

$$\hat{c}_{t+1} = p[\beta\hat{k}_t - \hat{k}_{t+1}] = p[\beta - (\delta\beta)^{1/v}]\hat{k}_t \quad \text{for } t \geq 0. \tag{14}$$

Since $\beta > (\delta\beta)^{1/v}$, $\hat{c}_{t+1}$ also increases (exponentially) to infinity (as $t \to \infty$), at the rate $(\delta\beta)^{1/v}$.

To summarize informally: *when* $\beta > h(b)$ *the optimal policy for the open economy is not to use the domestic investment good industry at all* ($\hat{x}_t = 0$ *for all* t) *no matter what* $\boldsymbol{k} \in (0, \bar{k})$ *is. The optimal capital stock and consumption grow exponentially over time, at the rate given by* (13).

Case (ii): $\beta < h(b)$

Here the constrained maximization problem (R′) is somewhat more subtle. The optimal x_t will typically depend on k_t. To see this, let us solve the problem (R): for any fixed $k > 0$

$$\begin{aligned} &\text{Maximize } [f(x) - \beta x] \\ &\text{subject to } 0 \leq x \leq k. \end{aligned} \tag{R}$$

Given our assumptions, there is a uniquely defined value $A > 0$, such that for $0 < x < A$, $f(x) - \beta x < 0$, and $f(A) - \beta A = 0$ (see Figure 2). Thus, if $k \in [0, A)$, problem (R) is solved uniquely by setting $x = 0$. When $k = A$, setting $x = 0$ or $x = k$ solves problem (R).

There is a uniquely defined value $B > 0$ such that $B > b$ and $f'(B) = \beta$ (see Figure 2). For $A < x < B$, $f'(x) > \beta$ and so $[f(x) - \beta x]$ is strictly increasing in x. Hence, for $k \in (A, B)$, problem (R) is solved uniquely by setting $x = k$. For $x > B$, $f'(x) < \beta$ and so $[f(x) - \beta x]$ is strictly decreasing in x. Thus, for $k \geq B$, problem (R) is solved uniquely by setting $x = B$.

Summarizing the above discussion, we can define a function, ϕ, as follows:

$$\phi(k) = \begin{cases} 0 & \text{if } k \in [0, A] \\ f(k) - \beta k & \text{if } k \in (A, B) \\ f(B) - \beta B & \text{if } k \geq B \end{cases} \tag{15}$$

Note that ϕ is the "value function" associated with the problem (R). That is,

$$\begin{aligned} \phi(k) = \ &\text{Max}\,[f(x) - \beta x] \\ &\text{subject to } 0 \leq x \leq k. \end{aligned}$$

Thus, the original dynamic optimization problem can be reduced to the following problem (using the function ϕ):

$$\begin{aligned} &\text{Maximize} && \sum_{t=0}^{\infty} \delta^t[\alpha k_t + p\phi(k_t) - pk_{t+1}]^{1-v} \\ &\text{subject to} && k_0 = \boldsymbol{k} \\ &&& \alpha k_t + p\phi(k_t) \geq pk_{t+1} \quad \text{for } t \geq 0. \end{aligned} \tag{16}$$

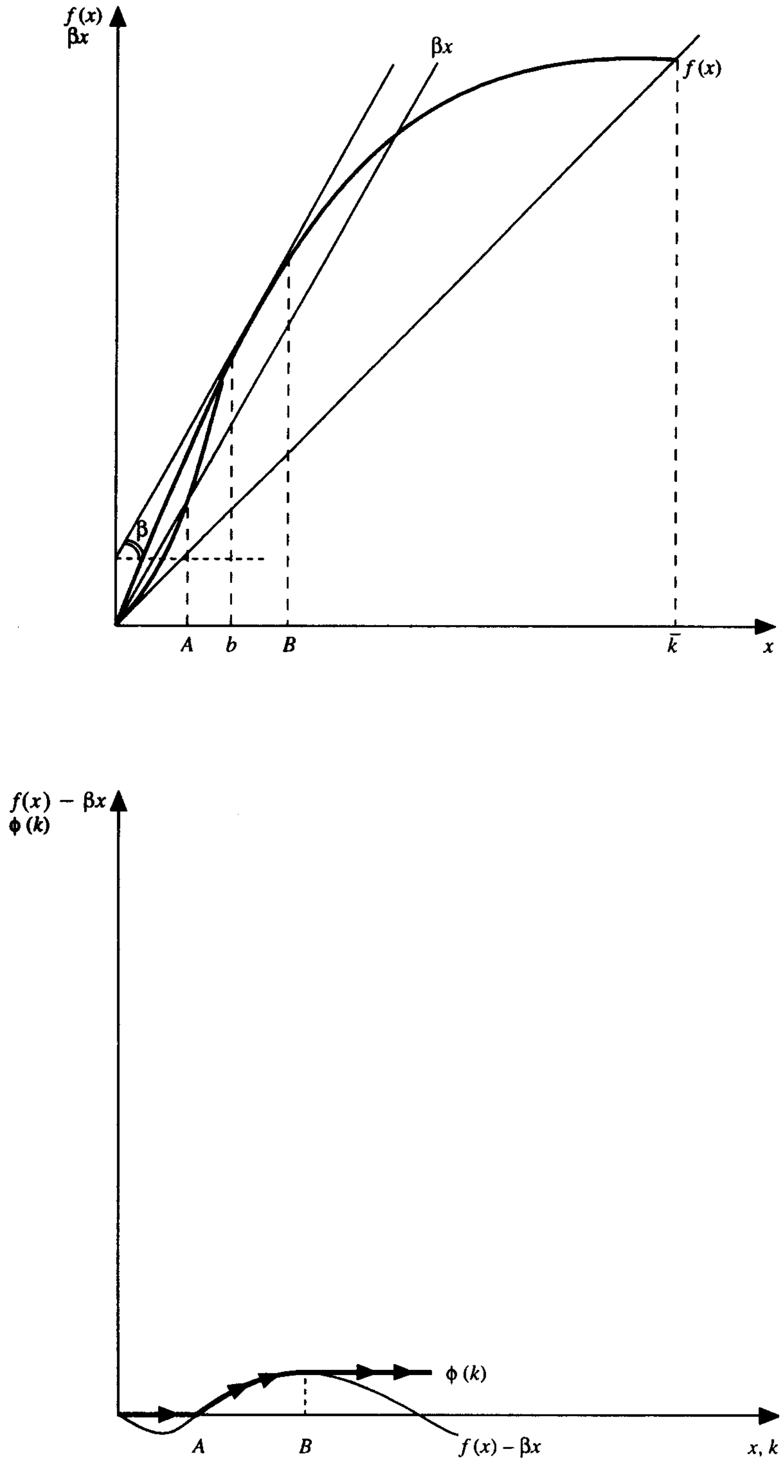


FIGURE 2.

This, in turn, reduces to the following dynamic optimization problem:

$$\begin{aligned} &\text{Maximize} && \sum_{t=0}^{\infty} \delta^t[\beta k_t + \phi(k_t) - k_{t+1}]^{1-v} \\ &\text{subject to} && k_0 = \boldsymbol{k} \\ &&& \beta k_t + \phi(k_t) \geqslant k_{t+1} \quad \text{for } t \geqslant 0. \end{aligned} \tag{17}$$

Defining a function ψ: $\Re_+ \to \Re_+$ by

$$\psi(k) = \phi(k) + \beta k. \tag{18}$$

we note that

$$\psi(k) = \begin{cases} \beta k & \text{if } k \in [0, A] \\ f(k) & \text{if } k \in (A, B) \\ f(B) - \beta B + \beta k & \text{if } k \geqslant B \end{cases} \tag{19}$$

Thus, $\psi(0) = 0$, and ψ is an increasing, continuous function on $\Re_+$ (see Figure 3). In terms of the function, ψ, we have the following dynamic optimization problem:

$$\begin{aligned} &\text{Maximize} && \sum_{t=0}^{\infty} \delta^t[\psi(k_t) - k_{t+1}]^{1-v} \\ &\text{subject to} && k_0 = \boldsymbol{k} \\ &&& \psi(k_t) \geqslant k_{t+1} \quad \text{for } t \geqslant 0. \end{aligned} \tag{P$''$}$$

This last problem (P$''$) is a familiar one. It is mathematically equivalent to an optimal growth problem for a non-trading economy, which has a production function, ψ, a utility function c^{1-v}, and a discount factor, δ.

To examine the nature of solutions to (P$''$), denote $[\psi(k_t) - k_{t+1}]$ by m_{t+1} for $t \geqslant 0$, and define the function $u(m) = m^{1-v}$ for $m \geqslant 0$.

Let $\{k_t\}$ be an optimal solution to (P$''$) from $\boldsymbol{k} > 0$. Then $m_{t+1} > 0$ and $K_t > 0$ for $t \geqslant 0$. Notice that $\psi(k)$ is continuously differentiable at all $k \geqslant 0$ except at $k = A$. At $k = A$, ψ does have a left-hand derivative (namely β), and a right-hand derivative (namely $f'(A)$). Clearly, $\beta = h(A) < f'(A)$.

We can now show that the optimal solution $\{k_t\}$ must satisfy $k_t \neq A$ for every $t \geqslant 1$. For, suppose $k_t = A$ for some $t \geqslant 1$. Then, (using the fact that $m_t > 0$ and $m_{t+1} > 0$), we can get $\varepsilon_t > 0$ such that for all $0 < \varepsilon < \varepsilon_t$,

$$u(\psi(k_{t-1}) - (k_t - \varepsilon)) + \delta u(\psi(k_t - \varepsilon) - k_{t+1}) \leqslant u(\psi(k_{t-1}) - k_t) + \delta u(\psi(k_t) - k_{t+1}) \tag{20}$$

and

$$u(\psi(k_{t-1}) - (k_t + \varepsilon)) + \delta u(\psi(k_t + \varepsilon) - k_{t+1}) \leqslant u(\psi(k_{t-1}) - k_t) + \delta u(\psi(k_t) - k_{t+1}). \tag{21}$$

Letting $\varepsilon \to 0$, we get from (20)

$$u'(m_t) + \delta u'(m_{t+1})\beta(-1) \leqslant 0 \tag{22}$$

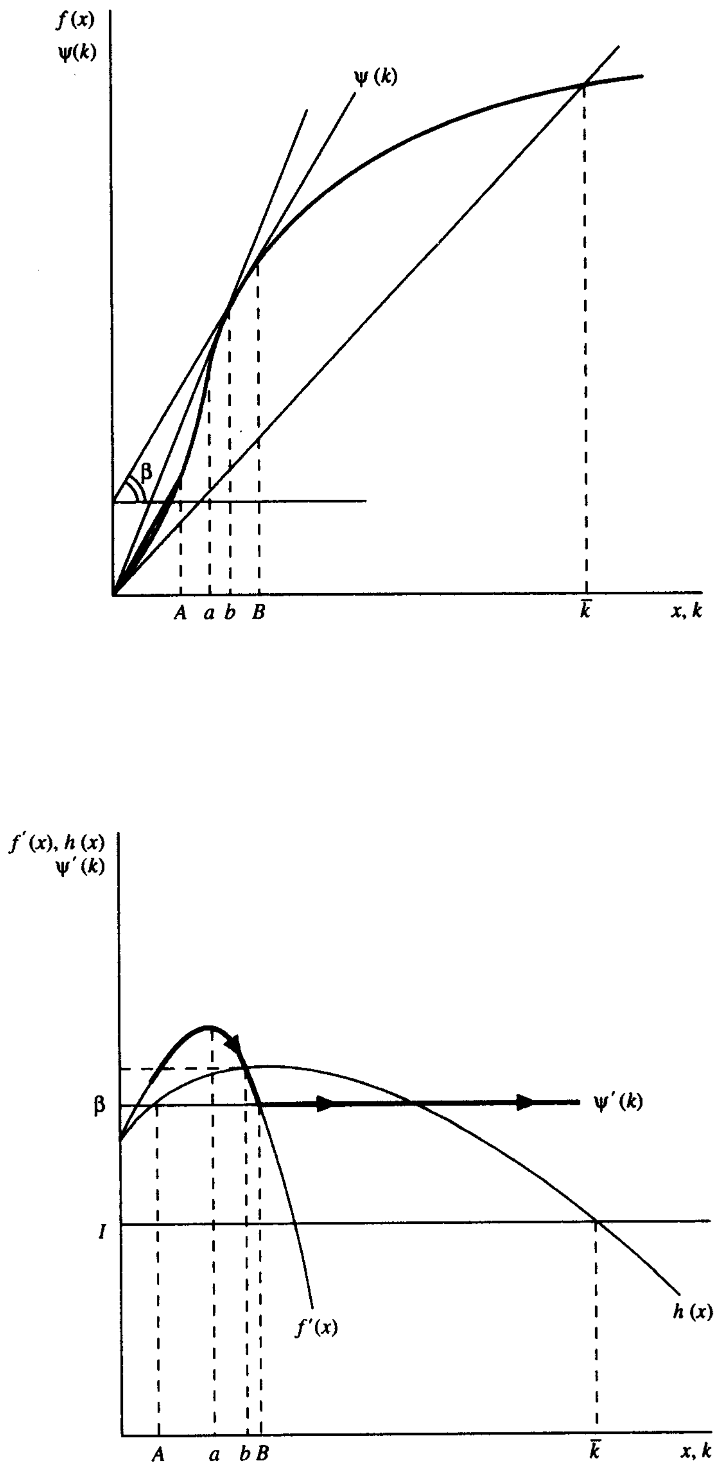


FIGURE 3.

and from (21)

$$u'(m_t)(-1) + \delta u'(m_{t+1})f'(A) \leqslant 0 \tag{23}$$

Combining (22) and (23)

$$\delta u'(m_{t+1})f'(A) \leqslant u'(m_t) \leqslant \delta u'(m_{t+1})\beta \tag{24}$$

which contradicts the fact that $f'(A) > \beta$.

Since $k_t \neq A$, and ψ is continuously differentiable at all $k \neq A$, we can follow the method in Theorem 1 and Corollary 1 in Dechert and Nishimura (1983) to conclude that either (i) $k_t \leqslant k_{t+1}$ for all $t \geqslant 0$, or (ii) $k_t \geqslant k_{t+1}$ for all $t \geqslant 0$.

We now show that case (ii) cannot occur. For if case (ii) occurred, then k_t converges either (a) to the origin, or (b) to a non-trivial stationary optimal stock. In subcase (a), there is T such that for all $t \geqslant T$, $k_t < A$ and so $\delta\psi'(k_t) = \delta\beta > 1$. Then, using the Ramsey-Euler equation, $u'(m_t) = u'(m_{t+1})\delta\psi'(k_t) > u'(m_{t+1})$, so that $m_t < m_{t+1}$ for $t \geqslant T$. In particular, $m_t \geqslant m_T > 0$ for all $t > T$. No programme for which k_t converges to zero can have m_t bounded away from zero, so subcase (a) is ruled out.

In subcase (b), if $\hat{k}$ is the non-trivial stationary optimal stock, then clearly $\hat{k}$ is not equal to A, as we have established above. Thus, by the Ramsey-Euler equations, we must have $\delta\psi'(\hat{k}) = 1$ (since $\psi(k) > k$ for all $k > 0$). But, for all $k \neq A$, we have $\delta\psi'(k) \geqslant \delta\beta > 1$. Thus, subcase (b) is also ruled out.

We have now established that $k_{t+1} \geqslant k_t$ for all $t \geqslant 0$. In fact, we must have $k_t \to \infty$ as $t \to \infty$. Otherwise, k_t would converge to a non-trivial stationary optimal stock, which can be ruled out by the argument used in the immediately preceding paragraph.

To summarize: *for any optimal programme from initial* $k > 0$, *the sequences* (k_t) *and* (c_t) *of optimal capitals and consumptions increase to infinity over time. The optimal allocation of capital between the two sectors is specified by (15). The exact formula for allocation depends on the total capital k and there are three stages of which two involve complete specialization in the production of a single good. For a "sufficiently large"* k ($\geqslant B$), *both domestic sectors are operated.*

Case (iii): $\beta = h(b)$

This is a "knife-edge" possibility which we discuss only briefly. Here, $x_t = 0$ is a solution to the constrained optimization problem (R′): in fact, for $k_t < b$, it is the *only* solution. However, when $k_t \geqslant b$, $x_t = b$ is also a solution. If $x_t = 0$ is chosen for all t, the analysis of production developed above for case (i) applies word for word. When $k_t \geqslant b$, we can complete our analysis by combining and modifying some of the arguments from the two cases spelled out above. Observe that, when $k_t \geqslant b$, even if $x_t = b$ is chosen the optimization problem (P) is still reduced to the problem (P′) [since $h(b) - \beta = 0$, use (10)]. The formulae (12)–(14) are still valid for describing the optimal policy and long run growth rates. Of course, if $x_t = b$, both the sectors are being used. Given that this case arises out of a "matching" of the parameters, we shall not dwell on the details of this case in our analysis of the pattern of trade that follows.

5. The effect of growth on the pattern of trade

In the discussion above, we have already noted that the condition $\beta > h(b)$ leads to the case of a complete specialization in the production of the consumption good. Hence, all along its path of development, the economy exports the consumption good and imports the investment good.

Consider now the condition $\beta < h(b)$. The dynamic optimization problem is (P″), and as noted above, for any *optimal* programme, the sequence (k_t) is monotonically increasing and is unbounded over time. Let the country be "capital poor" to begin with, i.e., $k_0 = \mathbf{k} > 0$ is sufficiently small. Since any optimal k_t from $\mathbf{k}$ goes to infinity (as t goes to infinity), we observe three phases of capital accumulation: (i) $0 < k_t \leqslant A$; (ii) $A < k_t < B$; (iii) $k_t \geqslant B$. We analyse the pattern of trade in each of the three phases.

In phase (i), as we observed in Section 4, $x_t = 0$ and the domestic investment-good industry is not used at all. Thus, the consumption good is exported and the investment good is imported in this phase.

As k_t increases beyond A, one enters phase (ii), and here $x_t = k_t$, so the consumption-good industry is not used at all. The investment good is exported and the consumption good is imported, exactly reversing the pattern of trade in phase (i).

In phase (iii), as k_t increases beyond B, x_t becomes fixed at the level B, and so the domestic investment good industry produces $f(B)$. The consumption-good industry is also used, and it produces $\alpha(k_t - B)$. The description of the pattern of trade in this phase requires some additional analysis which we now provide.

Define the function F: $\Re_+ \to \Re_+$ as follows:

$$F(k) = \begin{cases} h(b)k & \text{for } 0 \leqslant k \leqslant b \\ \psi(k) & \text{for } k \geqslant b \end{cases}$$

Note that $F(0) = 0$, and F is an increasing, concave and continuously differentiable function on $\Re_+$ (see Figure 4). Since $h(b) > \beta$, we have $F(k) \geqslant \psi(k)$ for all $k \geqslant 0$.

Consider the abstract dynamic optimization problem:

$$\begin{aligned} \text{Maximize} \quad & \sum_{t=0}^{\infty} \delta^t [F(k_t) - k_{t+1}]^{1-v} \\ \text{subject to} \quad & k_0 = \mathbf{k} \\ & k_{t+1} \leqslant F(k_t) \quad \text{for } t \geqslant 0. \end{aligned} \qquad (\text{P}''')$$

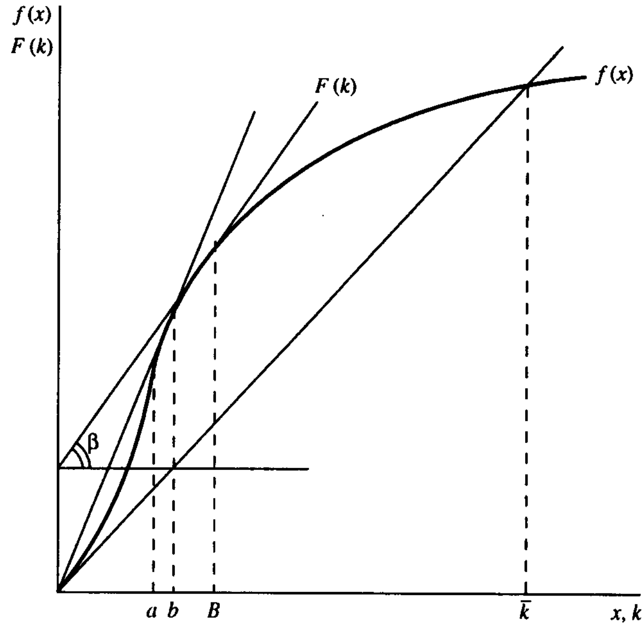
Clearly (P‴) is a dynamic optimization problem with a concave production function, F, a utility function c^{1-v} and a discount factor, δ. Thus, there is an optimal transition function, H: $\Re_+ \to \Re_+$, associated with (P‴), such that if $\{k_t\}$ solves (P‴), then

$$k_{t+1} = H(k_t) \quad \text{for } t \geqslant 0.$$

Furthermore, H is continuous on $\Re_+$.

Using the method of proof used in Theorem 1 of Dechert and Nishimura (1983), we can infer that H is increasing in k:

$$H(k') > H(k) \quad \text{if } k' > k$$



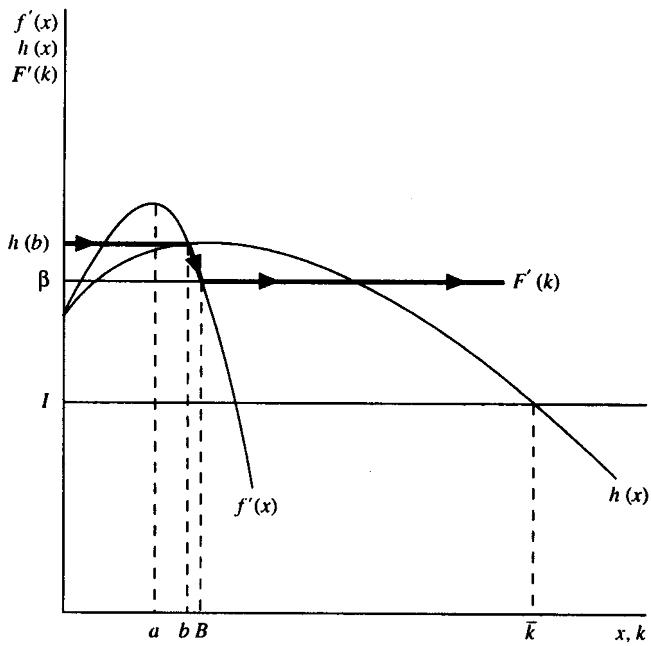


FIGURE 4.